Relativistic Spin-Flavor States in Light Front Dynamics

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Abstract

Orthonormal spin-flavor wave functions of Lorentz covariant quark models of the Bakamjian-Thomas type are constructed for nucleon resonances. Three different bases are presented. The manifestly Lorentz covariant Dirac-Melosh basis is related to the Pauli-Melosh basis and the symmetrized Bargmann-Wigner basis that are manifestly orthogonal.

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I. INTRODUCTION

The light-front form of dynamics introduced by Dirac [1] has by now become a powerful tool to treat relativistic multi-particle systems – as it provides a realization of the Poincaré Lie algebra with a maximal number of kinematical (interaction free) generators. In particular, the property that certain boosts are free of interactions is very appealing. For this reason, wave functions of moving frames may be connected by purely kinematic boosts. Moreover, with a well defined Fock expansion, no square root (Hamiltonian) operators and a simple vacuum structure, light front dynamics represents a viable framework for relativistic many-body theories.

As bound systems of three valence quarks predominantly, baryons are particularly interesting relativistic few-body states. The attractive features of the front form have motivated many recent calculations of various form factors of hadrons where rigorous transformation properties of wave functions under boosts are essential. And, as the field is rapidly developing, different formulations of relativistic few-body wave functions have emerged. Particle physicists prefer constructing multi-quark wave functions of hadrons (or so-called Joffe currents in the context of QCD sum rule techniques) using Dirac's gamma matrices. We refer to this approach as the Dirac-Melosh basis in the following [2–6]. Several nuclear physicists are using Melosh rotated nonrelativistic quark model (NQM) wave functions [7,8] which we call the Pauli-Melosh basis below. For the special case of ground state baryons the spin-isospin structure of three-quark wave functions can also be rigorously derived from group theoretical arguments [9]. At present there is little communication between these groups, though, which is evident even from some review articles [8,10]. Also for this reason we wish to provide a bridge between these different bases for baryons. We shall discuss their advantages and disadvantages and construct one basis from another. Finally, we present one basis that is useful but hardly known.

Here we do not address any form factor calculations nor related technical problems such as spurious parts of form factors caused by a lack of independence from the choice of light cone axis, angular conditions, Z-diagrams, etc. The question as to what dynamical equations are to be solved is not explored either. Except for the Pauli-Melosh approach that uses relativistic wave functions that are generated from a Schrödinger equation, this issue is complicated by the confinement problem of QCD, and no unambiguous solution is known as yet. The other bases allow one to go beyond the ladder-type Weinberg equation to include field theoretic effects such as Z-diagrams that are important in form factors and decays, but cannot be expressed in terms of wave functions alone. Our main goal is to explain the general relationship between major ingredients and methods of construction of different bases and clarify their relationships rather than give a complete review of all states. Such bases are expected to play a role in many non-perturbative QCD approaches.

We start from general kinematics in Sect.II and reconstruct the light cone momentum variables and Melosh rotations from the by now standard infinite momentum frame limit. Since the Pauli-Melosh basis has been extensively reviewed [8] we only give a terse description of it in Sect.III, while the Dirac-Melosh basis is discussed in greater detail in Sect.IV, as is the symmetrized Bargmann-Wigner basis in Sect.V along with its connection to the Dirac-Melosh basis in Sect.VI.

II. KINEMATICS

It was first shown by Susskind [11] that the infinite momentum frame (IMF) limit is equivalent to a change of the usual variables (t, x, y, z) into the light cone variables $(\tau^+ = t + z, x, y, \tau^- = t - z)$. This is demonstrated in the next subsection and used to derive the Melosh rotation that transforms the Dirac spinors into light cone spinors.

A. IMF and Light Cone Variables

Assume the observer moves with a large negative velocity along the z-axis relative to the baryon rest frame. In the observer's rest frame the baryon has the energy E and the momentum $P^\mu = (E, 0, 0, P)$. A quark has four-momentum $p^\mu = (p_0, \mathbf{p}_\perp, p_z)$, where $\mathbf{p}_\perp = (p_x, p_y)$ and is given through the boost $L_f(\omega_p)^\mu{}_\nu \, \mathring{p}^\nu$ with $\mathring{p}^\nu = (m, 0, 0, 0)$. The quark four-momentum in the baryon rest frame is denoted by k^μ and obtained by boosting along the z axis according to $k^\mu = L_c^{-1}(\omega_P)^\mu{}_\nu p^\nu$, or explicitly

$$\begin{aligned} k_0 &= p_0 \cosh\omega - p_z \sinh\omega, \\ k_z &= p_z \cosh\omega - p_0 \sinh\omega, \\ \mathbf{k}_\perp &= \mathbf{p}_\perp, \end{aligned} \tag{1}$$

where $\cosh\omega$ and $\sinh\omega$ are given by

$$\cosh\omega = \frac{E}{M_0}, \quad \sinh\omega = \frac{P}{M_0}, \tag{2}$$

and M_0 is the free invariant mass of the three quark system, $M_0^2 = P_\mu P^\mu$, given more explicitly later. In the infinite momentum frame (IMF), where $P \to \infty$, the longitudinal quark momentum p_z defines the momentum fraction η as

$$p_z = \eta P, \quad \lim P \to \infty. \tag{3}$$

In this limit each quark has a positive z-component so that $\eta > 0$ and $\sum_i \eta_i = 1$ due to momentum conservation. We will now derive the infinite momentum boost given by $\lim\limits_{P\to\infty} L_c^{-1}(\omega_P) L_f(\omega_p)$. Expanding the quark energy p_0 given by

$$p_0 = \sqrt{\eta^2 P^2 + \mathbf{p}_\perp^2 + m^2}, \tag{4}$$

in powers of $1/P$ we obtain in the infinite momentum limit

$$p_0 = \eta P + \frac{\mathbf{p}_\perp^2 + m^2}{2\eta P} + O(P^{-2}). \tag{5}$$

Similarly, the baryon energy is expanded as

$$E = P + \frac{M_0^2}{2P} + O(P^{-2}). \tag{6}$$

Substituting these relations Eqs. (5), (6) valid for $P \to \infty$ into Eq. (1) yields finite values for k_0 and k_z, the momentum in the nucleon rest frame,

$$\begin{aligned} k_0 &= \frac{1}{2}\left(\eta M_0 + \frac{\mathbf{p}_\perp^2 + m^2}{\eta M_0}\right), \\ k_z &= \frac{1}{2}\left(\eta M_0 - \frac{\mathbf{p}_\perp^2 + m^2}{\eta M_0}\right), \\ \mathbf{k}_\perp &= \mathbf{p}_\perp. \end{aligned} \tag{7}$$

Eq. (7) suggests introducing the light cone momentum components

$$\left(k^+ = k_0 + k_z,\ k^- = k_0 - k_z,\ \mathbf{k}_\perp\right),$$

if we identify the longitudinal fraction $\eta = p_z/P$ with the (kinematically invariant) light cone momentum fraction $x = p^+/P^+ = k^+/M_0$. To see this invariance of x, note that for any pure boost $L_c(\omega)$ in the z-direction, p^+ and p^- scale according to

$$\begin{aligned} L_c(\omega) p^\pm &= L_c(\omega)\left(p_0 \pm p_z\right) \\ &= \left(\cosh\omega \pm \sinh\omega\right) p^\pm = \exp\left(\pm\omega\right) p^\pm. \end{aligned} \tag{8}$$

Thus, Eq. (7) becomes a simple transformation of momentum variables [11,12]

$$\begin{aligned} k_0 &= \frac{1}{2}\left(x M_0 + \frac{\mathbf{k}_\perp^2 + m^2}{x M_0}\right) = \frac{1}{2}\left(k^+ + k^-\right), \\ k_z &= \frac{1}{2}\left(x M_0 - \frac{\mathbf{k}_\perp^2 + m^2}{x M_0}\right) = \frac{1}{2}\left(k^+ - k^-\right), \\ \mathbf{k}_\perp &= \mathbf{p}_\perp, \end{aligned} \tag{9}$$

since the invariant quark mass m and the light-cone energy variable k^- are related by

$$m^2 = k^+ k^- - \mathbf{k}_\perp^2, \quad k^- = \frac{\mathbf{k}_\perp^2 + m^2}{k^+}. \tag{10}$$

The scalar product for light cone coordinates is

$$a \cdot b = a_\mu b^\mu = \frac{1}{2}\left(a^+ b^- + a^- b^+\right) - \mathbf{a}_\perp \mathbf{b}_\perp, \tag{11}$$

For later purpose we will generalize Eq. (8) to arbitrary boost directions $\boldsymbol{\omega}$. For a moving quark with momentum p^μ the transformed components are given by

$$\begin{aligned} p'^+ &= p^+ \exp(\omega_z), \\ \boldsymbol{p}'_\perp &= \boldsymbol{p}_\perp + \boldsymbol{\omega}_\perp p^+ \exp(\omega_z). \end{aligned} \tag{12}$$

Eq. (9) is the main result of this paragraph. It defines a boost from the rest system of a quark to the rest system of the nucleon given in light cone coordinates, viz. $k^\mu = L_{cf}(\omega_k)^\mu{}_\nu \mathring{p}^\nu$. To find the proper transformation of the spinors we need the $SL(2,C)$ representation of L_{cf} which will be derived in the next subsection.

B. The Melosh Rotation

The proper transformation of the instant form spin states $|\vec{k}s'\rangle_{\rm inst}$ induced by the transformation to the light cone is given by

$$\begin{aligned} |k^+, \mathbf{k}_\perp s\rangle &= \sum R_{s's}\, |\vec{k}, s'\rangle_{\rm inst}, \\ |\vec{k}s\rangle_{\rm inst} &= \sum R^{-1}_{s's}\, |k^+, \mathbf{k}_\perp s'\rangle, \end{aligned} \tag{13}$$

where $R_{s's}$ is the Melosh rotation. We now derive the explicit form of the Melosh rotation matrix for Dirac spinors. To do so, we use the above demonstrated equivalence between the light cone frame and the infinite momentum frame. We will summarize this subsection in terms of a more elegant $SL(2,C)$ approach.

The instant-form Dirac spinor is written in the representation given, e. g., in Ref. [14],

$$u(\vec{p}, s) = \sqrt{\frac{p_0 + m}{2m}} \begin{pmatrix} \chi(s) \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi(s) \end{pmatrix}, \tag{14}$$

from which the light cone spinor u_{LC} is obtained by a boost to the IMF (where p depends on P, see Eqs. (3),(4)) and replacing $\eta \to x$ as explained in the previous subsection

$$u_{LC}\left(k^+,\mathbf{k}_\perp,s\right) = \lim_{P\to\infty} u(\vec{p},s), \tag{15}$$

with the spinor transformation matrix (corresponding to the boost in the negative z-direction)

$$S\,(k\leftarrow p) = \cosh\frac{\omega}{2} - \alpha_3 \sinh\frac{\omega}{2}. \tag{16}$$

The explicit form of u_{LC} then is

$$u_{LC}\left(k^+,\mathbf{k}_\perp,s\right) = \lim_{P\to\infty}\frac{1}{\sqrt{2m\,(p_0+m)}}\begin{pmatrix}\left[\cosh\frac{\omega}{2}\ (p_0+m) - \sinh\frac{\omega}{2}\,\sigma_3\vec{\sigma}\cdot\vec{p}\right]\chi(s)\\ \\ \left[\cosh\frac{\omega}{2}\,\vec{\sigma}\cdot\vec{p} - \sinh\frac{\omega}{2}\ (p_0+m)\,\sigma_3\right]\chi(s)\end{pmatrix},$$

upon using Eq. (2),

$$\cosh\frac{\omega}{2} = \sqrt{\frac{E+M_0}{2M_0}},\quad \sinh\frac{\omega}{2} = \sqrt{\frac{E+M_0}{2M_0}}\frac{P}{E+M_0} \simeq \sqrt{\frac{E+M_0}{2M_0}}\left(1-\frac{M_0}{P}\right)$$

$$p_0 = \eta P + \frac{\mathbf{p}_\perp^2+m^2}{2\eta P}. \tag{17}$$

The last approximation holds neglecting $O(P^{-2})$. Hence we obtain for u_{LC}

$$u_{LC}\left(k^+,\mathbf{k}_\perp,s\right) = \lim_{P\to\infty}\frac{1}{\sqrt{2m\,(p_0+m)}}\sqrt{\frac{E+M_0}{2M_0}}$$

$$\times\begin{pmatrix}\left[\eta P + \frac{\mathbf{p}_\perp^2+m^2}{2\eta P} + m - (-\boldsymbol{\sigma}_\perp\cdot\mathbf{p}_\perp\sigma_3 + \eta P)\left(1-\frac{M_0}{P}\right)\right]\chi(s)\\ \\ \left[\boldsymbol{\sigma}_\perp\cdot\mathbf{p}_\perp + \eta P\sigma_3 - \left(\eta P + \frac{\mathbf{p}_\perp^2+m^2}{2\eta P} + m\right)\sigma_3\left(1-\frac{M_0}{P}\right)\right]\chi(s)\end{pmatrix}. \tag{18}$$

Here the leading orders in P cancel and the remaining expression is independent of the momentum P, viz.

$$\begin{aligned}u_{LC}\left(k^+,\mathbf{k}_\perp,s\right) &= \frac{1}{2\sqrt{m\eta M_0}}\begin{pmatrix}[m+\eta M_0 + \boldsymbol{\sigma}_\perp\cdot\mathbf{k}_\perp\sigma_3]\,\chi(s)\\ \\ [\boldsymbol{\sigma}_\perp\cdot\mathbf{k}_\perp - m\sigma_3 + \eta M_0]\,\chi(s)\end{pmatrix}\\ &= \frac{1}{2\sqrt{mk^+}}\begin{pmatrix}(k^++m)\,\chi(s) + \boldsymbol{\sigma}_\perp\cdot\mathbf{k}_\perp\sigma_3\chi(s)\\ \\ (k^+-m)\,\sigma_3\chi(s) + \boldsymbol{\sigma}_\perp\cdot\mathbf{k}_\perp\chi(s)\end{pmatrix}.\end{aligned} \tag{19}$$

Introducing the standard form of the Pauli spinors [14] finally leads to

$$
\begin{aligned}
u_{LC}\left(k^+,\mathbf{k}_\perp,\uparrow\right) &= \frac{1}{2\sqrt{mk^+}}\begin{pmatrix} k^+ + m \\ k^R \\ k^+ - m \\ k^R \end{pmatrix}, \\
u_{LC}\left(k^+,\mathbf{k}_\perp,\downarrow\right) &= \frac{1}{2\sqrt{mk^+}}\begin{pmatrix} -k^L \\ k^+ + m \\ k^L \\ -k^+ + m \end{pmatrix},
\end{aligned} \tag{20}
$$

where $k^{L,R} = k_x \mp i k_y$. Similarly for the v-spinors

$$
\begin{aligned}
v_{LC}\left(k^+,\mathbf{k}_\perp,\uparrow\right) &= \frac{1}{2\sqrt{mk^+}}\begin{pmatrix} k^+ - m \\ k^R \\ k^+ + m \\ k^R \end{pmatrix}, \\
v_{LC}\left(k^+,\mathbf{k}_\perp,\downarrow\right) &= \frac{1}{2\sqrt{mk^+}}\begin{pmatrix} k^L \\ -k^+ + m \\ -k^L \\ k^+ + m \end{pmatrix}.
\end{aligned} \tag{21}
$$

For the Dirac spinor the Melosh rotation $R_{s's}$ defined in Eq. (13) is thus given by

$$
\begin{aligned}
u\left(\vec{k},\uparrow\right) &= \frac{1}{\sqrt{2k^+\left(k_0+m\right)}}\left[\left(k^+ + m\right)u_{LC}\left(k^+,\mathbf{k}_\perp,\uparrow\right) - k^R u_{LC}\left(k^+,\mathbf{k}_\perp,\downarrow\right)\right], \\
u\left(\vec{k},\downarrow\right) &= \frac{1}{\sqrt{2k^+\left(k_0+m\right)}}\left[\left(k^+ + m\right)u_{LC}\left(k^+,\mathbf{k}_\perp,\downarrow\right) + k^L u_{LC}\left(k^+,\mathbf{k}_\perp,\uparrow\right)\right].
\end{aligned} \tag{22}
$$

If we combine spin $\uparrow$ and spin $\downarrow$ into the fundamental representation of $SU(2)$, the matrix representation of R is given as

$$
(\uparrow,\downarrow)_{\rm inst} = (\uparrow,\downarrow)_{\rm LC}\begin{pmatrix} k^+ + m & k^L \\ -k^R & k^+ + m \end{pmatrix}\frac{1}{\sqrt{2k^+(m+k_0)}}, \tag{23}
$$

or

$$
R = \frac{(k^+ + m) + \boldsymbol{\sigma}_\perp\cdot\mathbf{k}_\perp\sigma_3}{\sqrt{2k^+(m+k_0)}}, \tag{24}
$$

which coincides with Refs. [7,8].

Alternatively, we now calculate the light cone spinor $u_{LC}(k^+, \mathbf{k}_\perp, s)$ given in Eq. (19) using the well known relation of proper Lorentz transformations to the $SL(2,C)$ covering group (see e.g. [13]). The respective $SL(2,C)$ representation of the IMF boost, Eq. (9), is given by

$$\sigma_\mu k^\mu = A\,\sigma_\mu \overset{\circ}{p}{}^\mu\, A^\dagger, \tag{25}$$

where $\sigma^\mu = (1, \sigma^i)$. The matrix $A \in SL(2,C)$ corresponding to the transformation given in Eq. (9) takes the form

$$A = \frac{1}{\sqrt{k^+ m}} \begin{pmatrix} k^+ & 0 \\ k_R & m \end{pmatrix}. \tag{26}$$

The corresponding transformation of Dirac spinors to the light cone that is required to give a complete representation of the full Lorentz group is given by [13]

$$S_{LC} = U \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} U^\dagger, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{27}$$

where U transforms to the diagonal Weyl representation of S_{LC}. The explicit form using Eq. (26) is then given by

$$S_{LC}(k) = \frac{1}{2\sqrt{mk^+}} \begin{pmatrix} (k^+ + m) + \boldsymbol{\sigma}_\perp \cdot \mathbf{k}_\perp \sigma_3 & (k^+ - m)\,\sigma_3 + \boldsymbol{\sigma}_\perp \cdot \mathbf{k}_\perp \\ (k^+ - m)\,\sigma_3 + \boldsymbol{\sigma}_\perp \cdot \mathbf{k}_\perp & (k^+ + m) + \boldsymbol{\sigma}_\perp \cdot \mathbf{k}_\perp \sigma_3 \end{pmatrix}. \tag{28}$$

Using $u_{LC}(k^+, \mathbf{k}_\perp, s) = S_{LC}(k) u(\vec{0}, s)$ then leads to Eq. (19).

C. Three-quark coordinates

The quark momentum variables for baryons in an arbitrary frame are denoted by p_1, p_2, p_3, while those in the baryon rest frame are written as k_1, k_2, k_3, with $\sum_{i=1}^{3} \vec{k}_i = 0$. The relation between p_i and k_i are given by Eqs. (12), (9)

$$\begin{aligned} \mathbf{k}_{i,\perp} &= \mathbf{p}_{i,\perp} - x_i \mathbf{P}_\perp, \quad x_i = \frac{p_i^+}{P^+} \\ k_{i,z} &= k^+ - k_0 = x_i M_0 - \frac{P \cdot p}{M_0} \\ &= x_i M_0 - \left(\frac{m_i^2 + (\mathbf{p}_{i\perp} - x_i \mathbf{P}_\perp)^2}{2 x_i M_0} + \frac{x_i M_0}{2} \right), \end{aligned} \tag{29}$$

where P is the total free four-momentum of the three-quark system and M_0 the invariant mass squared

$$M_0^2 = \sum_i \frac{m_i^2 + \mathbf{k}_{i,\perp}^2}{x_i}. \tag{30}$$

For proper symmetrization in the quark indices and equal quark masses $m_i = m_q$ we adopt the standard normalized Lovelace coordinates

$$\begin{aligned} \vec{k}_\rho &= \frac{1}{\sqrt{2}} \left(\vec{k}_1 - \vec{k}_2 \right), \\ \vec{k}_\lambda &= \frac{1}{\sqrt{6}} \left(\vec{k}_1 + \vec{k}_2 - 2\vec{k}_3 \right). \end{aligned} \tag{31}$$

Often used relativistic alternatives are the relative 4-momentum (space-like Jacobi) variables

$$q_3 = \frac{x_2 p_1 - x_1 p_2}{x_1 + x_2}, \;\; Q_3 = (1 - x_3) p_3 - x_3 (p_1 + p_2), \tag{32}$$

for the $\perp$ and $+$ components, and q_1, Q_1, q_2, Q_2 from cyclic permutation of the indices. Note that $\mathbf{Q}_{i\perp} = \mathbf{k}_{i\perp}$ for i=1,2,3.

III. PAULI-MELOSH BASIS

We now present the three quark basis generated by rotating the nonrelativistic wave function to the light cone [8]. Single particle state vectors $|p^+, \mathbf{p}_\perp \lambda\rangle$ are labeled by the momentum $(p^+, \mathbf{p}_\perp)$ and spin projection λ written in light front coordinates so that the mass shell condition $p^- = (m^2 + \mathbf{p}_\perp^2)/p^+$ is satisfied. Under a light front boost $p'^\mu = L^\mu{}_\nu \, p^\nu$, the state vectors transform unitarily as $U(L)|p^+, \mathbf{p}_\perp \lambda\rangle = \sqrt{p'^+/p^+} \, |p'^+, \mathbf{p}'_\perp \lambda\rangle$, where the light front spin λ remains unchanged (no Wigner rotation). These states are related to those of the instant form, $|\vec{p} m_s\rangle$, by the Melosh rotation $R_{cf} = \underline{L}_c^{-1}(\vec{p}) \underline{L}_f(\vec{p})$ (where $\underline{L}$ denotes the $SL(2, C)$ representation of L) so that

$$|p^+, \mathbf{p}_\perp \lambda\rangle = \sum_{m_s} \sqrt{E(\vec{p})/p^+} |\vec{p} m_s\rangle D^{1/2}_{m_s \lambda}(R_{cf}),$$

which corresponds to Eq. (13). Baryon three-quark states $|j; \vec{P}\lambda\rangle$ with spin j, spin projection λ and momentum $\vec{P}$ are related to wave functions according to

$$\langle \vec{p}_1 \lambda_1 \vec{p}_2 \lambda_2 \vec{p}_3 \lambda_3 | j; \vec{P}\lambda\rangle = \left| \frac{\partial \left(\vec{p}_1, \vec{p}_2, \vec{p}_3 \right)}{\partial \left(\vec{P}, \vec{k}_\rho, \vec{k}_\lambda \right)} \right|^{-1/2} (2\pi)^3 \delta(\sum_i \vec{p}_i - \vec{P})$$

$$\times \sum_{m's} \langle \frac{1}{2} m_1 \frac{1}{2} m_2 | s_{12} m_{12} \rangle \langle s_{12} m_{12} \frac{1}{2} m_3 | s m_s \rangle \langle l_\rho m_\rho l_\lambda m_\lambda | L m_L \rangle \tag{33}$$

$$\times \langle L m_L s m_s | j m \rangle Y_{l_\rho m_\rho} \left(\hat{\mathbf{k}}_\rho \right) Y_{l_\lambda m_\lambda} \left(\hat{\mathbf{k}}_\lambda \right) \Phi(k_\lambda, k_\rho)$$
$$\times D^{1/2\dagger}_{m_1 \lambda_1}(R(k_1)) D^{1/2\dagger}_{m_2 \lambda_1}(R(k_2)) D^{1/2\dagger}_{m_3 \lambda_1}(R(k_3)), \tag{34}$$

with obvious notation for the $SU(2)$ Clebsch-Gordan coefficients. The Jacobian is

$$\left| \frac{\partial \left(\vec{p}_1, \vec{p}_2, \vec{p}_3 \right)}{\partial \left(\vec{P}, \vec{k}_\rho, \vec{k}_\lambda \right)} \right| = \frac{p_1^+ p_2^+ p_3^+ M_0}{E(\vec{k}_1) E(\vec{k}_2) E(\vec{k}_3) P^+}.$$

The totally symmetric momentum wave functions of the bound state, which are not shown above, are separately orthogonal. Because of the orthogonality of the Melosh rotations, this basis of wave functions is manifestly orthogonal, which is clearly an advantage of the Pauli-Melosh basis. In contrast, Lorentz invariance is not manifest. Note, however, that relativistic models of the Bakamjian-Thomas type, where an interaction that commutes with the total spin is added to M_0, are Lorentz invariant if their interactions are rotationally invariant in terms of 3-vector (momentum or coordinate, etc.) variables of the particles [7,8,15]. This method of dealing with an interaction differs from that of field-theoretic Lagrangians and affects the connection with Feynman (and light-cone time-ordered) diagrams. A closely related light-front field theory [16] maintains the connection with light-cone time-ordered diagrams with interactions that do not necessarily commute with the total spin; it has different off-shell properties. A Dirac-Melosh basis is also constructed there that includes some states from configuration mixing but not all. Despite the three-vector appearance in Eq. (34), the quark momentum variables are relativistic, their z-components being defined in Eq. (9) in terms of their longitudinal momentum fraction x_i of Eq. (29) and M_0 of Eq. (30). The quark light-cone Pauli spinors depend on the relativistic quark momentum variables via the Melosh transformation. Therefore, despite working only with Pauli spinors (and coupling them by $SU(2)$ Clebsch-Gordan coefficients as in Eq. (34)), the Pauli-Melosh basis provides a consistent relativistic many-body framework for baryons. Small Dirac components are not necessary in such theories. In this sense, then, the Pauli-Melosh states form a minimal relativistic basis that is viable as long as pre-existing (or intrinsic) quark-antiquark excitations in baryons may be safely neglected at the low-energy scale Λ_{QCD}. As we shall see below, the Dirac-Melosh basis contains many-body states with small Dirac components in addition to the Pauli-Melosh basis.

In the baryon rest frame, the Pauli-Melosh basis becomes the usual NQM basis in the nonrelativistic limit, where the Melosh rotation is replaced by unity. Of course, this basis is far from complete as a relativistic Fock state basis. Only kinematic relativistic effects are included and boosts in particular. Dynamic (or intrinsic) quark-antiquark Fock components are not included but can be added in terms of the basis states of the Dirac-Melosh type shown for the nucleon in Table I and for nucleon resonances in Tables II and III in the next Sect. IV.

IV. BARYON SPIN-ISOSPIN STATES IN THE DIRAC-MELOSH BASIS

The NQM basis of hadron wave functions can be mapped one-to-one onto relativistic multi-quark light cone states. The first step in the construction of the Dirac-Melosh basis from the NQM basis consists in transforming the Clebsch-Gordan coefficients of the $SU(2)$ group into products of spin-isospin (flavor) matrix elements between quark and total momentum light cone spinors using the Wigner-Eckart theorem. Thereby, momentum dependence enters into the relativistic spin-flavor wave functions which, in the NQM, do not depend on momentum, thus removing manifest orthogonality from the Dirac-Melosh basis. A few examples of nucleon resonance states will serve us to illustrate this part of our procedure, e.g., the $N(938)$ is introduced in the next subsection, and subsequently the $N^*_{\frac{1}{2}^-}(1535), N^*_{\frac{3}{2}^-}(1520), N^*_{\frac{5}{2}^-}(1675)$.

A. The nucleon spin-isospin states

The Wigner-Eckart theorem allows one to rewrite the Clebsch-Gordan coefficients in the nonrelativistic nucleon spin-isospin wave function $|(s_{12}\frac{1}{2})J = \frac{1}{2}\ M_J = \lambda\rangle \times |(t_{12}\frac{1}{2})T = \frac{1}{2}\ M_T\rangle$ in terms of two products between Pauli spinors [2–4]

$$
\begin{aligned}
|(0\tfrac{1}{2})\tfrac{1}{2}\lambda\rangle &= \sqrt{\frac{1}{2}} \sum_{m_1 m_2 m_3} \left(\chi^\dagger_{m_1} i\sigma_2 \chi^*_{m_2}\right) \left(\chi^\dagger_{m_3}\chi_\lambda\right) \chi_{m_1}\chi_{m_2}\chi_{m_3}, \\
|(1\tfrac{1}{2})\tfrac{1}{2}\lambda\rangle &= -\sqrt{\frac{1}{6}} \sum_{m_1 m_2 m_3} \left(\chi^\dagger_{m_1} \vec{\sigma} i\sigma_2 \chi^*_{m_2}\right) \cdot \left(\chi^\dagger_{m_3}\vec{\sigma}\chi_\lambda\right) \chi_{m_1}\chi_{m_2}\chi_{m_3},
\end{aligned} \tag{35}
$$

The associated isospin matrix elements have the same form as Eq. (35). Under a Melosh rotation of the Pauli spinors, Eq. (13), to the light cone, Eqs. (35) retain their form. After introducing light cone spinors this equation then matches the spin part of the nucleon wave function of the Pauli-Melosh basis given in the previous section. The generalization of Eqs. (35) to covariant expressions will now be achieved in two further steps. First, we express the Pauli light cone spinors by the Dirac light cone spinors given in the nucleon rest system. In a second step this will be generalized to arbitrary nucleon momenta.

The light cone Pauli spinor is given by the upper part of the light cone Dirac spinor (just as it is the case for the instant form Dirac spinor). This may be formally written as a projection, viz.

$$
\chi_{LC} = N\ [1,0]\ u_{LC}(k), \tag{36}
$$

where $[1,0]$ is understood as a 2×4 matrix written in block notation. Also a normalization factor $N = \sqrt{2m/(k_0+m)}$ has been introduced to respect proper normalization. Therefore, we are able to replace the matrix products between Pauli spinors by products between light cone Dirac spinors. As an illustration we write

$$\chi^{\dagger}_{LC,m_1} i\sigma_2 \chi^{*}_{LC,m_2} = N_1 N_2 \, \bar{u}_{LC}(k_1) \, [1,0]^{\top} i\sigma_2 [1,0] \, \bar{u}^{\top}_{LC}(k_2), \tag{37}$$

where $N_1 = N(k_1)$. Expressing the matrix element in terms of standard Dirac matrices leads to

$$[1,0]^{\top} i\sigma_2 [1,0] = \frac{1}{2}(1+\gamma_0)\gamma_5 i\gamma_0\gamma_2(1+\gamma_0). \tag{38}$$

The other matrix elements are rewritten in a similar way.

To write the resulting expressions in an arbitrary frame moving with the nucleon momentum $(P^+, \mathbf{P}_\perp)$, i.e. for each quark $p^\mu = L(\omega_P)^\mu{}_\nu k^\nu$, note that each quark spinor experiences a transformation according to

$$\bar{u}_{LC}(k) = \bar{u}_{LC}(p) \, \bar{S}(\omega_P). \tag{39}$$

The 4×4 matrix $\bar{S}(\omega_P) = \gamma_0 S^{+}(\omega_P)\gamma_0$ may be written in the following way (compare Ref. [14] for instant form spinors)

$$\bar{S}(w_P) = \left[u^{\uparrow}_{LC}(P), u^{\downarrow}_{LC}(P), v^{\uparrow}_{LC}(P), v^{\downarrow}_{LC}(P) \right]. \tag{40}$$

With Eqs. (39) and (40) we can write the expressions given in Eqs. (37), (38) in an invariant form

$$\begin{aligned} & \bar{u}_{LC}(k_1) \, (1+\gamma_0) \, \gamma_5 i\gamma_0\gamma_2 \, (1+\gamma_0) \, \bar{u}^{\top}_{LC}(k_2) \\ &= \frac{1}{M_0^2} \bar{u}_{LC}(p_1) \, (\gamma \cdot P + M_0) \, \gamma_5 C \left(\gamma^{\top} \cdot P + M_0 \right) \bar{u}^{\top}_{LC}(p_2) \\ &= \frac{2}{M_0} \bar{u}_{LC}(p_1) \, (\gamma \cdot P + M_0) \, \gamma_5 C \bar{u}^{\top}_{LC}(p_2), \end{aligned} \tag{41}$$

where $C = i\gamma_0\gamma_2$ is the charge conjugation matrix.

From inspection, we recognize that the invariant expression may be derived from the nonrelativistic one by simply replacing the Pauli spinors in the matrix elements of Eq. (35) via

$$\chi_{LC} \to N \, (\gamma \cdot P + M_0) u_{LC}(k^+, \mathbf{k}_\perp, \lambda), \qquad N = 2\sqrt{\frac{mx}{(xM_0+m)^2 + \mathbf{k}_\perp^2}}, \tag{42}$$

where M_0^2 is the free mass squared of the three-quark system of Eq. (30) and P the free total momentum of the system in Lorentz covariant Bakamjian-Thomas models [15].

In a last step, we can combine the spin invariants shown in Eq. (35) with the corresponding isospin matrix elements so as to obtain 1↔2 symmetric invariants which are then symmetrized in the *uds* basis. All steps combined yield the relativistic spin-isospin wave function of the nucleon ψ_N in the covariant form with $G = i\tau_2 C$,

$$\psi_N = \mathcal{N} \sum_{\lambda_i} \Big[\Big(\bar{u}_1(\gamma \cdot P + M_0)\gamma_5 G \bar{u}_2^\top \Big) (\bar{u}_3 u_\lambda) + (23)1 + (31)2 \Big] u_1 u_2 u_3, \tag{43}$$

where the two factors $(\gamma \cdot P + M_0)$ in the (12) matrix element are combined into one factor as in Eq. (41) and the third projection factor in the second matrix element $(\bar{u}_3 u_\lambda)$ is eliminated by using the Dirac equation for the nucleon. In Eq. (43), the $\bar{u}_i$ and u_i are abbreviations of $\bar{u}_{LC}(p_i, \lambda_i), u_{LC}(p_i, \lambda_i)$, and the normalization $\mathcal{N}$ that includes the normalization factors from Eq. (42) among others; it determines the charge form factor of the proton at zero momentum transfer.

A complete set of relativistic spin-isospin invariants of the form Eq. (41) is given in Table I. The ⊗ symbolizes that each G_i consists of the product of two matrix elements between Dirac (light-cone) spinors, i.e. is of the form (12)3. The Dirac spinors for the quarks have been omitted for simplicity. These spin-isospin states are symmetrized in the *uds* basis [17], where quarks are treated as distinguishable; if the third quark is taken to be the down quark, then the up quarks are symmetrized explicitly in the spin-flavor wave function. That is why the isospin operator is chosen so as to make the (12) matrix element symmetric under 1 ↔ 2.

Clearly, the nucleon wave function of Eq. (43) is the symmetrized combination $(G_2 + G_6)$. From the construction it should be clear that the normalized wave function including the Melosh normalization factors of Eq. (42) coincides with the corresponding one from the Pauli-Melosh basis. In order to obtain a nucleon wave function component from the G_i of Table I the Dirac spinors must be included in each invariant as in Eq. (43) and symmetrized, i.e. adding $(23)1 + (31)2$ to (12)3. In order to generate a component corresponding to the Pauli-Melosh basis the projection onto Pauli spinors via $(\gamma \cdot P + M_0)$ for each quark is required as well. For G_1, the projection factors from quark 1 and 2 yield $(\gamma \cdot P + M_0)(-\gamma \cdot P + M_0) = 0$. When the projections are similarly applied to the mixed symmetric spin-isospin invariants G_3 and G_8, the same channel wave function ψ_N is generated. Just like G_1, G_5 and G_7 make no contribution to ψ_N either. Hence, just as the nonrelativistic nucleon spin-isopin wave function is unique, so is its relativistic generalization of the Pauli-Melosh basis [2,4,5]. This is also the case for other baryon wave functions. These procedures apply similarly to all relativistic spin-flavor wave functions for nucleon resonances that will be discussed in the following. Let us note, however, that mesonic or sea quark-antiquark Fock components will lead to new spin-flavor wave functions and configuration mixing expands the nucleon basis by N* states with the spin-flavor quantum numbers of the nucleon.

B. The $N^*_{\frac{1}{2}^-}(1535)$ spin-isospin states

We start again from the nonrelativistic states for the P-wave given below. The notation is $|\left[(s_{12}\frac{1}{2})\frac{1}{2}L\right] JM_J\rangle$, where $L=1$ in this case. The corresponding Dirac-Melosh state has been constructed in [5]. The Clebsch-Gordan coefficients in the nonrelativistic state are written as products of matrix elements as for the nucleon so that

$$
\begin{aligned}
|\left[(0\tfrac{1}{2})\tfrac{1}{2}1\right] \tfrac{1}{2}\lambda\rangle &= -\sqrt{\frac{1}{8\pi}} \sum_{m_1m_2m_3} \left(\chi^\dagger_{m_1} i\sigma_2 \chi^*_{m_2}\right) \left(\chi^\dagger_{m_3} \vec{\sigma}\cdot(\vec{k}_1-\vec{k}_2)\ \chi_\lambda\right) \chi_{m_1}\chi_{m_2}\chi_{m_3}, \\
|\left[(1\tfrac{1}{2})\tfrac{1}{2}1\right] \tfrac{1}{2}\lambda\rangle &= \sqrt{\frac{1}{24\pi}} \sum_{m_1m_2m_3} \left(\chi^\dagger_{m_1} \vec{\sigma} i\sigma_2 \chi^*_{m_2}\right) \cdot \left(\chi^\dagger_{m_3} \vec{\sigma}\ \vec{\sigma}\cdot(\vec{k}_1-\vec{k}_2)\ \chi_\lambda\right) \chi_{m_1}\chi_{m_2}\chi_{m_3}.
\end{aligned} \quad (44)
$$

These states differ from the nonrelativistic nucleon states only by the P-wave factor $\vec{\sigma}\cdot(\vec{k}_1-\vec{k}_2)$ so that the relativistic states are directly obtained from the nucleon states by the replacement $u_N \to \gamma_5\,\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_{N^*}$ in Eq. (43) and Table I. The four-vector $\tilde{p}_1-\tilde{p}_2=(p_1-p_2)-((p_1-p_2)\cdot P/P^2)P$ reduces to $\vec{k}_1-\vec{k}_2$ in the baryon rest frame, as it is the case in the orbital wave functions of the Pauli-Melosh basis. The use of $\tilde{p}_1-\tilde{p}_2$ guarantees the orthogonality of different orbital angular momentum states in the Dirac-Melosh basis.

Again, the spin-isospin states will be symmetrized in the *uds* basis, where quarks are treated as distinguishable and light quarks are symmetrized explicitly in the spin-flavor wave function. Therefore, it suffices to consider the mixed antisymmetric relative momentum p_1-p_2 of Eq. (31) since the mixed symmetric $p_1+p_2-2p_3$ terms will be automatically generated by symmetrizing. The resulting relativistic expression for the spin-isospin wave function is then

$$
\psi_{N^*} = \mathcal{N}\sum_{\lambda_i}\left[\left(\bar{u}_1(\gamma\cdot P+M_0)\gamma_5\vec{\tau}G\bar{u}_2^\top\right)\left(\bar{u}_3\gamma_5\,\gamma\cdot(\tilde{p}_1-\tilde{p}_2)\vec{\tau}u_\lambda\right)+(23)1+(31)2\right]u_1u_2u_3, \quad (45)
$$

for the dominant $N^*_{\frac{1}{2}^-}(1535)$ configuration. The $(\gamma\cdot P+M_0)$ factor from the third quark has been removed using the commutator $[\gamma\cdot P, \gamma\cdot(p_1-p_2)]$ and $P\cdot(\tilde{p}_1-\tilde{p}_2)=0$. Again, this wave function coincides with the corresponding one from the Pauli-Melosh basis. The complete set of spin invariants is given in Table II.

C. $N^*_{\frac{3}{2}^-}(1520)$ Basis States

We now consider the $J=\frac{3}{2}^-$ resonance and therefore introduce Rarita-Schwinger spinors. In the LS-coupling scheme the nonrelativistic spin wave function of the $N^*_{\frac{3}{2}^-}(1520)$ of spin

$\frac{3}{2}$ and negative parity in the baryon rest frame is written again in terms of matrix elements between Pauli spinors using the Wigner-Eckart theorem

$$
\begin{aligned}
|\left[(0\tfrac{1}{2})\tfrac{1}{2}1\right]\tfrac{3}{2}\lambda\rangle &= -\sqrt{\frac{3}{8\pi}}\sum_{m_1m_2m_3}\left(\chi^\dagger_{m_1}i\sigma_2\chi^*_{m_2}\right)\left(\chi^\dagger_{m_3}\left(k_{1\,\nu}-k_{2\,\nu}\right)u^\nu_{\frac{3}{2}\lambda}\right)\chi_{m_1}\chi_{m_2}\chi_{m_3}, \\
|\left[(1\tfrac{1}{2})\tfrac{1}{2}1\right]\tfrac{3}{2}\lambda\rangle &= \sqrt{\frac{1}{8\pi}}\sum_{m_1m_2m_3}\left(\chi^\dagger_{m_1}\vec{\sigma}i\sigma_2\chi^*_{m_2}\right)\cdot\left(\chi^\dagger_{m_3}\vec{\sigma}\left(k_{1\,\nu}-k_{2\,\nu}\right)u^\nu_{\frac{3}{2}\lambda}\right)\chi_{m_1}\chi_{m_2}\chi_{m_3}.
\end{aligned}
\tag{46}
$$

Here we have used Rarita-Schwinger spinors to describe the spin $\frac{3}{2}$ state and introduced ϵ^μ via

$$\vec{\epsilon}_m\vec{k} = -\epsilon^\mu_m(\mathring{P})k_\mu, \tag{47}$$

in the rest system of the baryon, where $\mathring{P}^\mu = (M_0, \vec{0})$, and $\epsilon^\mu_m(\mathring{P})$ is the spin-1-vector [19]

$$\epsilon^\mu_m(\mathring{P}) = (0, \epsilon_m). \tag{48}$$

With Clebsch-Gordan coefficients the ϵ^μ_m and N^* Pauli spinor are combined to a Rarita-Schwinger spinor.

In a general frame, the polarization vectors $\epsilon^\mu_\lambda(P) = L_\nu{}^\mu(P)\epsilon^\nu_\lambda(\mathring{P})$ satisfy the orthogonality and normalization conditions

$$
\begin{aligned}
&P\cdot\epsilon_+ = P\cdot\epsilon_- = P\cdot\epsilon_0 = 0, \\
&\epsilon^\dagger_+\cdot\epsilon_+ = \epsilon^\dagger_-\cdot\epsilon_- = \epsilon^\dagger_0\cdot\epsilon_0 = -1, \\
&\epsilon^\dagger_+\cdot\epsilon_- = \epsilon^\dagger_+\cdot\epsilon_0 = \epsilon^\dagger_-\cdot\epsilon_0 = 0,
\end{aligned}
\tag{49}
$$

so that, more explicitly,

$$
\begin{aligned}
\epsilon^\mu_+ &= -\frac{1}{\sqrt{2}}\left(0, 1, i, 2P^R/P^+\right), \\
\epsilon^\mu_0 &= \frac{1}{M_0}\left(P^+, P^1, P^2, (\mathbf{P}^2_\perp - M^2_0)/P^+\right), \\
\epsilon^\mu_- &= \frac{1}{\sqrt{2}}\left(0, 1, -i, 2(P^L)/P^+\right).
\end{aligned}
\tag{50}
$$

Again due to proper orthogonality we use the formally covariant expression $\tilde{p}$ for $\vec{k}$ in the nucleon rest frame. These spin wave functions can be written as Lorentz covariant expressions

$$I_0 = \left(\bar{u}_1 \gamma_5 \left(\gamma \cdot P + M_0\right) C \bar{u}_2^\top\right) \left(\bar{u}_3 (\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^\nu_{\frac{3}{2}\lambda}\right),$$
$$I_1 = \left(\bar{u}_1 \left(\gamma \cdot P + M_0\right) \gamma^\mu C \bar{u}_2^\top\right) \left(\bar{u}_3 \gamma_\mu \gamma_5 (\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^\nu_{\frac{3}{2}\lambda}\right). \quad (51)$$

The construction of these basis states closely follows the rules outlined in the previous sections. In particular, in order to construct a P-wave state with orbital angular momentum $L = 1$, the Dirac-Melosh states should contain one momentum $\tilde{p}$ factor. In addition, in the nonrelativistic limit all basis states should either be linear combinations of I_0 or I_2, or vanish. A set of basis states is then given by substituting $u \to \tilde{p}_\nu u^\nu_{\frac{3}{2}\lambda}$ in the nucleon basis, where $u^\nu_{\frac{3}{2}\lambda}$ is the Rarita-Schwinger spinor using the uds basis with $p \equiv p_\rho$. The construction of further invariants is restricted by the following constraints of the Rarita-Schwinger spinors

$$\gamma_\mu u^\mu_{\frac{3}{2}\lambda} = 0, \quad P_\mu u^\mu_{\frac{3}{2}\lambda} = 0, \quad (52)$$

that lead to

$$P_\mu u^\mu_{\frac{3}{2}\lambda} = \frac{1}{2} \gamma_\mu \gamma \cdot P u^\mu_{\frac{3}{2}\lambda}. \quad (53)$$

Hence the associated properly symmetrized wave function may be written as

$$\psi_{N^*} = \mathcal{N} \sum_{\lambda_i} \left[\bar{u}_1 (\gamma \cdot P + M_0) \gamma_5 \vec{\tau} G \bar{u}_2^\top \bar{u}_3 \vec{\tau} (\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^\nu_{\frac{3}{2}\lambda} + (23)1 + (31)2\right] u_1 u_2 u_3. \quad (54)$$

The isospin dependence is determined so as to give symmetric combinations for the $1 \leftrightarrow 2$ matrix element. This wave function can be constructed directly from the invariants in Table III, as explained for the nucleon case.

D. Basis States for $N^*_{\frac{5}{2}^-}(1675)$

The nonrelativistic representation of the state $|\frac{3}{2} \times 1\rangle^{\frac{5}{2}}$ is given by

$$\left|\left[(1\tfrac{1}{2})\tfrac{3}{2}1\right] \tfrac{5}{2}\lambda\right\rangle = \sqrt{\frac{3}{8\pi}} \sum_{m_1 m_2 m_3} \left(\chi^\dagger_{m_1} \sigma_\mu i \sigma_2 \chi^*_{m_2}\right) \left(\chi^\dagger_{m_3} \left(\tilde{p}_{1\,\nu} - \tilde{p}_{2\,\nu}\right) u^{\mu\nu}_{\frac{5}{2}\lambda}\right), \quad (55)$$

using the formally covariant representation $\vec{k} \to \tilde{p}$. In the uds basis the momentum is chosen to be the mixed antisymmetric combination $(\tilde{p}_{1\,\nu} - \tilde{p}_{2\,\nu})$. Invariants which contain $p_\nu u^{\mu\nu}$ are, for example,

$$
\begin{aligned}
I_0 &= \bar{u}_1 M_0 \gamma_\mu G \bar{u}_2^\top \bar{u}_3 (\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^{\mu\nu}_{\frac{5}{2}\lambda} + (23)1 + (31)2, \\
I_1 &= \bar{u}_1 M_0 \gamma_\mu \gamma_5 G \bar{u}_2^\top \bar{u}_3 (\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^{\mu\nu}_{\frac{5}{2}\lambda} + (23)1 + (31)2, \\
I_2 &= \bar{u}_1 i M_0 \sigma_{\rho\nu} G \bar{u}_2^\top \bar{u}_3 \gamma^\rho (\tilde{p}_{1\mu} - \tilde{p}_{2\mu}) u^{\mu\nu}_{\frac{5}{2}\lambda} + (23)1 + (31)2, \\
I_3 &= \bar{u}_1 i \sigma_{\rho\nu} P^\nu G \bar{u}_2^\top \bar{u}_3 (\tilde{p}_{1\mu} - \tilde{p}_{2\mu}) u^{\mu\rho}_{\frac{5}{2}\lambda} + (23)1 + (31)2. \qquad (56)
\end{aligned}
$$

Further invariants are restricted by the Rarita-Schwinger constraint given in Eq. (52). The corresponding wave function obtained from the Melosh rotated NQM state takes the standard form

$$
\psi_{N^*} = \mathcal{N} \sum_{\lambda_i} \left[\bar{u}_1 (\gamma \cdot P + M_0) \gamma_5 G \bar{u}_2^\top \bar{u}_3 (\tilde{p}_{1\mu} - \tilde{p}_{2\mu})(\tilde{p}_{1\nu} - \tilde{p}_{2\nu}) u^{\mu\nu}_{\frac{5}{2}\lambda} \; + (23)1 + (31)2 \right] u_1 u_2 u_3. \quad (57)
$$

Summarizing, the advantages of the Dirac-Melosh basis are the ease and transparency of (e.g. current) matrix element calculations and the manifest (kinematic) rotational and Lorentz transformation properties of the wave functions which follow from the use of free light-cone Dirac spinors for the quarks and total momentum motion. Amongst its disadvantages are the need of Fierz transformations if one wants to rewrite the (23)1 and (31)2 permutations of wave function components in the canonical 123 quark order. These are avoided in the Bargmann-Wigner basis to be discussed in the following Sects.V and VI.

V. SYMMETRIZED BW BASIS

The general Bargmann-Wigner (BW) basis [9,20,21] of relativistic three-particle states contains 64 product states of three free particle light cone spinors U^λ, V^λ that satisfy the free Dirac equations

$$
(\gamma \cdot P - M_0)\, U^\lambda(P) = 0, \qquad (58)
$$

$$
(\gamma \cdot P + M_0)\, V^\lambda(P) = 0, \qquad (59)
$$

where P^μ is total (free particle) baryon momentum and M_0 its mass.

Restricting this basis to definite parity and positive total spin projection S_z obviously leaves the 16 states B_1 to B_{16} that are shown in Table IV.

The BW basis has several advantages. For one, the nonrelativistic limit is obtained just by deleting the V spinors, and the extreme relativistic limit, $(P_z \to \infty)$, by setting $U^\uparrow = V^\uparrow$ and $U^\downarrow = -V^\downarrow$.

Moreover, product states with particle permutations are readily expressed in the BW basis, whereas the Dirac-Melosh basis requires Fierz transformations [3] when particle indices are interchanged. Its disadvantage that spins are not well defined is removed by symmetrizing the product states appropriately, as will be shown next. To this end, U and V Dirac spinors are combined to form the fundamental representation of a $SU(2)_R$ so that the spin-isospin wave function of a baryon is represented as

$$SU(2)_S \otimes SU(2)_R \otimes SU(3)_F,$$

and each state has well defined spin and permutation symmetry in the first two particles.

For three quarks the representations of $SU_S(2) \otimes SU_R(2) \subseteq SU(4)$ are displayed in the Table V, where the functions ξ and φ for different permutation symmetries (S, A, M_S, M_A) are defined in Table VI and Table VII. We denote completely symmetric $SU(2)_R \otimes SU(2)_S$ states by S, primed for spin $\frac{3}{2}$ and unprimed for spin $\frac{1}{2}$, mixed symmetric by s, mixed antisymmetric by a and completely antisymmetric by A.

To further simplify the notation we use φ' for $m_S = +\frac{3}{2}$ states and φ for $m_S = +\frac{1}{2}$ states, ξ' for $m_R = 3$ and ξ for $m_R = -1$ positive parity states. The functions $S_1', ..., A$ not shown in Table V are given in terms of product functions $\xi\varphi$ in Table VIII.

For the nucleon we get three different basis states to construct a mixed symmetric basis with $S = \frac{1}{2}$ $m_s = \frac{1}{2}$,

$$\begin{aligned} \xi_S' \varphi_{M_S} &= |UUU\rangle \times |\frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \uparrow\rangle = \frac{1}{\sqrt{2}} \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow \right) U^\uparrow, \\ \xi_S \varphi_{M_S} &= |\frac{1}{\sqrt{3}} (VVU + UVV + VUV)\rangle \times |\frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \uparrow\rangle, \\ (\xi_{M_S} \varphi_{M_S} - \xi_{M_A} \varphi_{M_A}) &= \frac{1}{\sqrt{18}} \Big[\left(V^\uparrow U^\downarrow + U^\downarrow V^\uparrow \right) V^\uparrow + \left(U^\uparrow V^\uparrow + V^\uparrow U^\uparrow \right) V^\downarrow \\ &\quad + \left(V^\uparrow V^\downarrow + V^\downarrow V^\uparrow \right) U^\uparrow \Big]. \end{aligned} \tag{60}$$

In the nonrelativistic case the V spinors vanish so that only the term with $m_R = 3$ survives. This way the spin wave function is defined with proper total spin and proper pair permutation symmetry. The spin $\frac{1}{2}$ states [9,21] are

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{2}} \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow \right) U^\uparrow \\ a_2 &= \frac{1}{\sqrt{6}} \left[\left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow \right) U^\uparrow + \left(U^\uparrow V^\downarrow - U^\downarrow V^\uparrow + V^\uparrow U^\downarrow - V^\downarrow U^\uparrow \right) V^\uparrow \right] \\ a_3 &= \frac{1}{\sqrt{6}} \left[- \left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow \right) U^\uparrow + \left(V^\uparrow U^\downarrow - U^\downarrow V^\uparrow \right) V^\uparrow + \left(U^\uparrow V^\uparrow - V^\uparrow U^\uparrow \right) V^\downarrow \right] \\ s_1 &= \frac{1}{\sqrt{6}} \left[\left(U^\uparrow U^\downarrow + U^\downarrow U^\uparrow \right) U^\uparrow - 2 U^\uparrow U^\uparrow U^\downarrow \right] \\ s_2 &= \frac{1}{\sqrt{18}} \left[\left(V^\uparrow U^\downarrow + V^\downarrow U^\uparrow + U^\uparrow V^\downarrow + U^\downarrow V^\uparrow \right) V^\uparrow \right] \end{aligned}$$

$$+\frac{1}{\sqrt{18}}\left[-2\left(U^\uparrow V^\uparrow+V^\uparrow U^\uparrow\right)V^\downarrow+\left(V^\uparrow V^\downarrow+V^\downarrow V^\uparrow\right)U^\uparrow-2V^\uparrow V^\uparrow U^\downarrow\right]$$
$$s_3=\frac{1}{\sqrt{18}}\left[\left(V^\uparrow U^\downarrow+U^\downarrow V^\uparrow\right)V^\uparrow+\left(U^\uparrow V^\uparrow+V^\uparrow U^\uparrow\right)V^\downarrow+\left(V^\uparrow V^\downarrow+V^\downarrow V^\uparrow\right)U^\uparrow\right]$$
$$+\frac{1}{\sqrt{18}}\left[-2\left(U^\uparrow V^\downarrow+V^\downarrow U^\uparrow\right)V^\uparrow-2V^\uparrow V^\uparrow U^\downarrow\right]$$
$$S=\frac{1}{\sqrt{18}}\left[\left(V^\uparrow V^\downarrow+V^\downarrow V^\uparrow\right)U^\uparrow+\left(U^\uparrow V^\downarrow+V^\downarrow U^\uparrow\right)V^\uparrow+\left(V^\uparrow U^\uparrow+U^\uparrow V^\uparrow\right)V^\downarrow\right]$$
$$+\frac{1}{\sqrt{18}}\left[-2U^\downarrow V^\uparrow V^\uparrow-2V^\uparrow U^\downarrow V^\uparrow-2V^\uparrow V^\uparrow U^\downarrow\right]$$
$$A=\frac{1}{\sqrt{6}}\left[\left(U^\uparrow V^\downarrow-V^\downarrow U^\uparrow\right)V^\uparrow-\left(V^\uparrow V^\downarrow-V^\downarrow V^\uparrow\right)U^\uparrow+\left(V^\uparrow U^\uparrow-U^\uparrow V^\uparrow\right)V^\downarrow\right], \tag{61}$$

and the spin $\frac{3}{2}$ states [9,21]

$$a_1'=\frac{1}{\sqrt{6}}\left[-\left(V^\uparrow U^\uparrow-U^\uparrow V^\uparrow\right)V^\downarrow+\left(V^\uparrow U^\downarrow-U^\uparrow V^\downarrow\right)V^\uparrow+\left(V^\downarrow U^\uparrow-U^\downarrow V^\uparrow\right)V^\uparrow\right]$$
$$a_2'=\frac{1}{\sqrt{2}}\left(V^\uparrow U^\uparrow-U^\uparrow V^\uparrow\right)V^\uparrow$$
$$s_1'=\frac{1}{\sqrt{18}}\left[\left(V^\uparrow U^\uparrow+U^\uparrow V^\uparrow\right)V^\downarrow+\left(V^\uparrow U^\downarrow+U^\uparrow V^\downarrow\right)V^\uparrow+\left(V^\downarrow U^\uparrow+U^\downarrow V^\uparrow\right)V^\uparrow\right]$$
$$+\frac{1}{\sqrt{18}}\left[-2V^\uparrow V^\uparrow U^\downarrow-2V^\uparrow V^\downarrow U^\uparrow-2V^\downarrow V^\uparrow U^\uparrow\right]$$
$$s_2'=\frac{1}{\sqrt{6}}\left[\left(V^\uparrow U^\uparrow+U^\uparrow V^\uparrow\right)V^\uparrow-2V^\uparrow V^\uparrow U^\uparrow\right]$$
$$S_1'=\frac{1}{\sqrt{3}}\left[U^\uparrow U^\uparrow U^\downarrow+U^\uparrow U^\downarrow U^\uparrow+U^\downarrow U^\uparrow U^\uparrow\right]$$
$$S_1''=U^\uparrow U^\uparrow U^\uparrow$$
$$S_2'=\frac{1}{3}\left[V^\uparrow V^\uparrow U^\downarrow+V^\uparrow V^\downarrow U^\uparrow+V^\downarrow V^\uparrow U^\uparrow+V^\uparrow U^\uparrow V^\downarrow+V^\uparrow U^\downarrow V^\uparrow+V^\downarrow U^\uparrow V^\uparrow\right]$$
$$+\frac{1}{3}\left[U^\uparrow V^\uparrow V^\downarrow+U^\uparrow V^\downarrow V^\uparrow+U^\downarrow V^\uparrow V^\uparrow\right]$$
$$S_2''=\frac{1}{\sqrt{3}}\left[V^\uparrow V^\uparrow U^\uparrow+V^\uparrow U^\uparrow V^\uparrow+U^\uparrow V^\uparrow V^\uparrow\right]. \tag{62}$$

The orbital function which have (S, M_A, M_S) symmetries is given by

$$\mathcal{Y}_{Lm_L}=\sum_{m_\rho m_\lambda}\langle \ell_\rho m_\rho l_\lambda m_\lambda|Lm_L\rangle Y_{\ell_\rho m_\rho}(\hat{\mathbf{p}}_\rho)Y_{l_\lambda m_\lambda}(\hat{\mathbf{p}}_\lambda)=\left[Y^{[\ell_\rho]}(\hat{\mathbf{p}}_\rho)\otimes Y^{[\ell_\lambda]}(\hat{\mathbf{p}}_\lambda)\right]^{[L]}_{m_L}. \tag{63}$$

As an example for the $L=1$ case we want to propose a construction method for BW basis states for the $N^*_{\frac{1}{2}^-}(1535)$ resonance. As in the previous subsection, we consider only the antisymmetric orbital function

$$\psi_{M_A} = k_\rho Y_{1m}(\hat{\mathbf{k}}_\rho) = \mathcal{Y}_{1m}(k_\rho)\,, \tag{64}$$

where k_ρ is the relative momentum in the nucleon rest frame denoted by k in the following for simplicity. Depending on the z-projection, ψ_{M_A} is a function of the rest frame momentum variables k_z, k_R or k_L

$$\mathcal{Y}_{10}(k) = \sqrt{\frac{3}{4\pi}}k_z, \quad \mathcal{Y}_{1\pm1}(k) = \mp\sqrt{\frac{3}{8\pi}}\,(k_x \pm ik_y)\,. \tag{65}$$

We generalize these momentum variables to a general frame using Eq. (29) in the following manner

$$k_0 = \frac{P\cdot p}{M_0},\; k_z = -\frac{P\cdot p}{M_0} + \frac{M_0 p^+}{P^+},\; k_R = p_R - \frac{P_R p^+}{P^+},\; k_L = p_L - \frac{P_L p^+}{P^+}. \tag{66}$$

Now the orbital function ψ_{M_A} can be coupled with positive parity antisymmetric spin $\frac{1}{2}$ functions $\chi^{\frac{1}{2}+}_{M_A} = a_1, a_2, a_3$ or symmetric functions $\chi^{\frac{1}{2}+}_{M_S} = s_1, s_2, s_3$ shown in Tables V and VIII, by means of Clebsch-Gordon coefficients

$$\begin{aligned}\left[\chi^{\frac{1}{2}+}_{M} \times \mathcal{Y}_1\right]^{\frac{1}{2}} &= \sqrt{\frac{2}{3}}\chi^{\frac{1}{2}+}_{M,+\frac{1}{2}}\mathcal{Y}_{11} - \sqrt{\frac{1}{3}}\chi^{\frac{1}{2}+}_{M,-\frac{1}{2}}\mathcal{Y}_{10} \\ = -\frac{1}{\sqrt{4\pi}}&\left[\left(p_R - \frac{P_R p^+}{P^+}\right)\chi^{\frac{1}{2}+}_{M,-\frac{1}{2}} + \left(p_L - \frac{P_L p^+}{P^+}\right)\chi^{\frac{1}{2}+}_{M,+\frac{1}{2}}\right].\end{aligned} \tag{67}$$

VI. TRANSFORMATION OF DIRAC-MELOSH INTO BW BASIS

We now expand the Dirac-Melosh states G_i of the nucleon basis in Table I (of Sect.IV) into the Bargmann-Wigner states B_i of Table IV (in the previous Sect.V). This will be done first for the nucleon and subsequently for the other nucleon resonances, where the abbreviation $G = Ci\tau_2$ is used.

A. The nucleon spin-isospin states

The second nucleon term $G_2 = M_0\gamma_5 Ci\tau_2 \otimes U$ in Eq. (43) is an even γ-matrix coupling U with U and V with V, combined with (12) antisymmetric spin structure. Therefore,

$$[\gamma_5 C \otimes U^\uparrow]_{123} = (UU+VV)U \times (\uparrow\downarrow - \downarrow\uparrow)\uparrow = \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow + V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right) U^\uparrow. \quad (68)$$

In the other nucleon term $G_6 = M_0\gamma_0\gamma_5 Ci\tau_2 \otimes U$ in the rest frame of the three-quark system in Eq. (43), γ_0 changes the sign of the VV term so that

$$[\gamma_0\gamma_5 C \otimes U^\uparrow]_{123} = (UU-VV)U \times (\uparrow\downarrow - \downarrow\uparrow)\uparrow . \quad (69)$$

In order to check, e.g. the coefficient of the $V^\uparrow V^\downarrow$ term in Eq. (68) we have to calculate the overlap matrix element

$$\bar{V}^\uparrow_\alpha \bar{V}^\downarrow_\beta (\gamma_5 C)_{\alpha\beta} = -\bar{V}^\uparrow_\alpha V^\uparrow_\alpha = \frac{1}{4M_0}\mathrm{Tr}[(1-\gamma_5\gamma\cdot\epsilon_0)(M_0 - \gamma\cdot P)] = 1, \quad (70)$$

using

$$\bar{V}^\downarrow_\beta(\gamma_5 C)_{\alpha\beta} = -V^\uparrow_\alpha,$$

The other terms in Eq. (68) can be checked similarly. Much the same reasoning yields for G_1 and G_5

$$\begin{aligned}[C \otimes \gamma_5 U^\uparrow]_{123} &= (UV+VU)V \times (\uparrow\downarrow - \downarrow\uparrow)\uparrow, \\ [\gamma_0 C \otimes \gamma_5 U^\uparrow]_{123} &= (UV-VU)V \times (\uparrow\downarrow - \downarrow\uparrow)\uparrow . \end{aligned} \quad (71)$$

These relations correspond to the spinor identities

$$\begin{aligned} C &= (UV+VU)\times(\uparrow\downarrow - \downarrow\uparrow) = U^\uparrow V^\downarrow - U^\downarrow V^\uparrow + V^\uparrow U^\downarrow - V^\downarrow U^\uparrow, \\ \gamma_5 C &= (UU+VV)\times(\uparrow\downarrow - \downarrow\uparrow) = U^\uparrow U^\downarrow - U^\downarrow U^\uparrow + V^\uparrow V^\downarrow - V^\downarrow V^\uparrow, \end{aligned}$$

$$\begin{aligned} (\gamma\cdot P + M_0)\,\gamma_5 C &= 2M_0 UU \times (\uparrow\downarrow - \downarrow\uparrow) = 2M_0\left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right), \\ (M_0 - \gamma\cdot P)\, C &= M_0(UV+VU)(\uparrow\downarrow - \downarrow\uparrow). \end{aligned} \quad (72)$$

The vector term G_3 has a more complicated spin-isospin structure [3]

$$\begin{aligned} &[\gamma^\mu C \otimes \gamma_5\gamma_\mu U^\uparrow]_{123} \\ &= (UU-VV)U \times (-2\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) + (UV-VU)V \times (\uparrow\downarrow - \downarrow\uparrow)\uparrow, \end{aligned} \quad (73)$$

etc. In summary, any Dirac–Melosh basis state of Table I may be written as a linear combination of Bargmann–Wigner basis states

$$G_i = \sum_{j=1,\ldots,16} c_j B_j, \quad (74)$$

using the above (Eq. (72)) and the following spinor identities

$$
\begin{aligned}
U^\uparrow \bar{U}^\uparrow &= \frac{1}{4M_0}(1+\gamma_5\gamma\cdot\epsilon_0)(M_0+\gamma\cdot P)\\
U^\downarrow \bar{U}^\downarrow &= \frac{1}{4M_0}(1-\gamma_5\gamma\cdot\epsilon_0)(M_0+\gamma\cdot P)\\
U^\uparrow \bar{U}^\downarrow &= \frac{\sqrt{2}}{4M_0}\gamma_5\gamma\cdot\epsilon_+(M_0+\gamma\cdot P)\\
U^\downarrow \bar{U}^\uparrow &= -\frac{\sqrt{2}}{4M_0}\gamma_5\gamma\cdot\epsilon_-(M_0+\gamma\cdot P)\\
U^\uparrow \bar{V}^\uparrow &= -\frac{1}{4M_0}(\gamma_5-\gamma\cdot\epsilon_0)(M_0-\gamma\cdot P)\\
U^\downarrow \bar{V}^\downarrow &= -\frac{1}{4M_0}(\gamma_5+\gamma\cdot\epsilon_0)(M_0-\gamma\cdot P)\\
U^\uparrow \bar{V}^\downarrow &= \frac{\sqrt{2}}{4M_0}\gamma\cdot\epsilon_+(M_0-\gamma\cdot P)\\
U^\downarrow \bar{V}^\uparrow &= -\frac{\sqrt{2}}{4M_0}\gamma\cdot\epsilon_-(M_0-\gamma\cdot P)\\
V^\uparrow \bar{U}^\uparrow &= \frac{1}{4M_0}(\gamma_5+\gamma\cdot\epsilon_0)(M_0+\gamma\cdot P)\\
V^\downarrow \bar{U}^\downarrow &= \frac{1}{4M_0}(\gamma_5-\gamma\cdot\epsilon_0)(M_0+\gamma\cdot P)\\
V^\uparrow \bar{U}^\downarrow &= \frac{\sqrt{2}}{4M_0}\gamma\cdot\epsilon_+(M_0+\gamma\cdot P)\\
V^\downarrow \bar{U}^\uparrow &= -\frac{\sqrt{2}}{4M_0}\gamma\cdot\epsilon_-(M_0+\gamma\cdot P)\\
V^\uparrow \bar{V}^\uparrow &= -\frac{1}{4M_0}(1-\gamma_5\gamma\cdot\epsilon_0)(M_0-\gamma\cdot P)\\
V^\downarrow \bar{V}^\downarrow &= -\frac{1}{4M_0}(1+\gamma_5\gamma\cdot\epsilon_0)(M_0-\gamma\cdot P)\\
V^\uparrow \bar{V}^\downarrow &= \frac{\sqrt{2}}{4M_0}\gamma_5\gamma\cdot\epsilon_+(M_0-\gamma\cdot P)\\
V^\downarrow \bar{V}^\uparrow &= -\frac{\sqrt{2}}{4M_0}\gamma_5\gamma\cdot\epsilon_-(M_0-\gamma\cdot P),
\end{aligned} \tag{75}
$$

where the polarization vectors [19] ϵ^μ_λ were given in Eq. (50).

It is instructive to note that the Melosh rotation can also be written as $\bar{u}_{\lambda_i}(p_i)U^\lambda(P)$ since

$$
\begin{aligned}
\bar{u}^\uparrow(p_i)U^\uparrow &= (p_i^+M_0+m_iP^+)/(2\sqrt{m_iM_0p_i^+P^+}),\\
\bar{u}^\uparrow(p_i)U^\downarrow &= (P^+p_i^L-P^Lp_i^+)/(2\sqrt{m_iM_0p_i^+P^+}),\\
\bar{u}^\downarrow(p_i)U^\uparrow &= (P^Rp_i^+-P^+p_i^R)/(2\sqrt{m_iM_0p_i^+P^+}).
\end{aligned} \tag{76}
$$

In order to illustrate the usefulness of the BW basis let us first consider the nucleon in the *uds* basis where quarks are treated as distinguishable. For example, if the d quark in the proton is taken as the third quark, then only the two up quarks are symmetrized in the spin-flavor wave function. Rewriting the proton wave function with the spin up mixed antisymmetric function $\chi^{\uparrow}_{M_A}$ and the mixed antisymmetric isospin part ϕ_{M_A} it becomes

$$\begin{aligned} 2\phi_{M_A}\chi^{\uparrow}_{M_A} &= (ud - du)u\,[(\uparrow\downarrow - \downarrow\uparrow)\uparrow \times UUU] \\ &= u_2 d_3 u_1\,[(\uparrow_2\downarrow_3 - \downarrow_2\uparrow_3)\uparrow_1 \times UUU] - d_3 u_1 u_2\,[(\uparrow_3\downarrow_1 - \downarrow_3\uparrow_1)\uparrow_2 \times UUU] \\ &= u_1 u_2 d_3\,[(\uparrow_1\uparrow_2\downarrow_3 - \uparrow_1\downarrow_2\uparrow_3 - \downarrow_1\uparrow_2\uparrow_3 + \uparrow_1\uparrow_2\downarrow_3) \times UUU] \\ &= uud\,[(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \times UUU] = -\sqrt{6}uud\chi^{\uparrow}_{M_S}, \end{aligned} \tag{77}$$

which obviously leads to a mixed symmetric component. Writing the relativistic proton wave function of Eq. (43) in the same 123 quark order of the BW basis starting from

$$[G_2 + G_6]_{123} = \left(U^{\uparrow}U^{\downarrow} - U^{\downarrow}U^{\uparrow}\right)U^{\uparrow},$$

now easily yields

$$[G_2 + G_6]_{231} = [\left(U^{\uparrow}U^{\downarrow} - U^{\downarrow}U^{\uparrow}\right)U^{\uparrow}]_{231} = U^{\uparrow}U^{\uparrow}U^{\downarrow} - U^{\uparrow}U^{\downarrow}U^{\uparrow}, \tag{78}$$

and hence

$$\begin{aligned} [G_2 + G_6]_{231} - [G_2 + G_6]_{312} &= U^{\uparrow}U^{\uparrow}U^{\downarrow} - U^{\uparrow}U^{\downarrow}U^{\uparrow} - U^{\downarrow}U^{\uparrow}U^{\uparrow} + U^{\uparrow}U^{\uparrow}U^{\downarrow} \\ &= 2U^{\uparrow}U^{\uparrow}U^{\downarrow} - U^{\uparrow}U^{\downarrow}U^{\uparrow} - U^{\downarrow}U^{\uparrow}U^{\uparrow}, \end{aligned} \tag{79}$$

the same result. The corresponding lines in the Dirac-Melosh basis involve the much more complicated Fierz transformations [3] to rewrite the (23)1 and (31)2 forms in the canonical 123 order of spin invariants, which are avoided here in the BW basis, which is a definite advantage.

B. The $N^*_{\frac{1}{2}^-}(1535)$ spin-isospin states

Now we consider the $N^*_{\frac{1}{2}^-}(1535)$ states of negative parity in the Dirac-Melosh basis, shown in Table II and relate them to the symmetrized BW basis states in Eq. (71) and Eq. (72). To that end we expand the $\gamma \cdot (p_1 - p_2)$ part, and here abbreviating $p_1 - p_2 = p_\rho = p$,

$$\begin{aligned} \gamma \cdot p\; U^{\uparrow} &= a_p U^{\uparrow} + c_p V^{\uparrow} + d_p V^{\downarrow} \\ \gamma \cdot p\; U^{\downarrow} &= a_p U^{\downarrow} + d^*_p V^{\uparrow} - c_p V^{\downarrow} \\ \gamma \cdot p\; V^{\uparrow} &= -c_p U^{\uparrow} - d_p U^{\downarrow} - a_p V^{\uparrow} \\ \gamma \cdot p\; V^{\downarrow} &= -d^*_p U^{\uparrow} + c_p U^{\downarrow} - a_p V^{\downarrow}, \end{aligned} \tag{80}$$

with the coefficients

$$
\begin{aligned}
a_p &= \frac{P \cdot p}{M_0} = k_0 \\
c_p &= \frac{M_0\, p^+}{P^+} - \frac{P \cdot p}{M_0} = k_z \\
d_p &= p^R - \frac{P^R p^+}{P^+} = k_R \\
d_p^* &= p^L - \frac{P^L p^+}{P^+} = k_L,
\end{aligned} \tag{81}
$$

where $p_L = p_x - ip_y, p_R = p_x + ip_y$ as defined before. Therefore the expanded states of Eq. (80) to the usual form without $\gamma \cdot p$ dependence can be straightforward mapped to the BW basis.

As an example we expand the first term of the wave function ψ_{N^*} given in Eq. (45) in the symmetrized BW basis. Using

$$
(\gamma \cdot P)\,(\gamma \cdot p) = P \cdot p - iP^\mu p^\nu \sigma_{\mu\nu} = 2P \cdot p - (\gamma \cdot p)\,(\gamma \cdot P)\,, \tag{82}
$$

we can write (again with $p = p_1 - p_2$)

$$
\begin{aligned}
&[(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[(\gamma \cdot P + M_0)\,\gamma \cdot p\gamma_5 U^\uparrow\right] \\
&= [(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[2M_0\gamma \cdot p\gamma_5 U^\uparrow\right] + [(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[2P \cdot p\gamma_5 U^\uparrow\right].
\end{aligned}
$$

Now we transform these terms to the symmetrized BW basis. We then get with the expression for $(\gamma \cdot P + M_0)\,\gamma_5 C$ given in Eq. (72) and for $\gamma \cdot p\gamma_5 U^\uparrow$ in Eq. (80)

$$
\begin{aligned}
&[(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[(\gamma \cdot P + M_0)\,\gamma \cdot p\;\gamma_5 U^\uparrow\right] \\
&= -2M_0\,[(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[\left(\frac{M_0\,p^+}{P^+} - \frac{P \cdot p}{M_0}\right) U^\uparrow + \left(p^R - \frac{P^R p^+}{P^+}\right) U^\downarrow + \frac{P \cdot p}{M_0} V^\uparrow\right] \\
&\quad + [(\gamma \cdot P + M_0)\,\gamma_5 C] \otimes \left[2P \cdot p\; V^\uparrow\right] \\
&= -4M_0^2 \left[\left(\frac{M_0\,p^+}{P^+} - \frac{P \cdot p}{M_0}\right) \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right) U^\uparrow \right. \\
&\quad \left. + \left(p^R - \frac{P^R p^+}{P^+}\right) \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right) U^\downarrow \;\right].
\end{aligned} \tag{83}
$$

We find that the wave function ψ_{N^*} given in Eq. (45) can be expressed in BW basis states shown in Eq. (67) with the particular choice $\chi = a_1$.

VII. CONCLUSION

We have reviewed several ways of constructing basis states for three valance quarks to describe baryons on the light cone. These are the Pauli-Melosh, the Dirac-Melosh, the Bargmann-Wigner and the symmetrized Bargmann-Wigner bases. All of these bases ensure that wave functions of moving frames are connected by purely kinematic boosts. We have compared these bases to each other and discussed their respective advantages.

The *Pauli-Melosh* basis is a minimal extension of the nonrelativistic basis to ensure proper kinematical boosts that connect moving frames. Due to the use of Pauli spinors that are properly Melosh rotated the nonrelativistic coupling scheme can be kept. Therefore incorporation of angular momentum is possible within the same algebra on the expense of manifest covariance.

From the construction of states in the *Dirac-Melosh* basis it is obvious that these basis states are relativistic generalizations of the Pauli-Melosh basis states as they ensure the full Dirac spinor structure. The *Dirac-Melosh* basis can be systematically enlarged to describe nucleon resonances with non-zero orbital angular momentum. As an example, the case for $\ell = 1$ basis states for $N^*(1535)$ and $N^*(1520)$ are shown. The Lorentz covariance of the Dirac-Melosh basis is manifest. However, the orthogonality of the basis states must be proved explicitly.

For the nucleon we have shown that the Dirac-Melosh basis can be mapped onto the *symmetrized Bargmann-Wigner* basis. Orthogonality and completeness of this basis are fulfilled by construction. The Melosh rotation is implicitly present through products $\bar{u}_{\lambda_i}(p_i)U^\lambda(P)$. The simple symmetrization procedure leads to advantages in practical calculations. We extended the *symmetrized Bargmann-Wigner* basis to states with angular momentum $\ell = 1$ and showed that the resulting basis states are equivalent to the Dirac-Melosh basis states.

As the Pauli-Melosh basis is using two component spinors, the extension of nonrelativistic codes using standard angular momentum decomposition should be straightforward at the expense mentioned previously in Section II B.

Many processes, in particular those involving transition amplitudes, require a covariant treatment of baryons. Even the calculation of elastic amplitudes involve moving frames (e.g., the Breit frame) to compare with experimental data, whereas wave functions are usually given in their respective rest frames. Therefore a manifestly covariant wave function is clearly appealing in cases were not only the wave functions in the rest frame of the three quark system is involved.

Also, the covariant bases potentially allow connecting wave functions given in the rest system to high energy physics phenomena and, therefore, investigating processes involved in the interesting transition regime from the broken chiral symmetry region to the gluon dominated region of quantum chromodynamics (accessible, e.g., at TJNAF).

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TABLES

TABLE I. Relativistic spin invariants for the nucleon N, $|(s_{12}\frac{1}{2});\frac{1}{2}^{+}\lambda\rangle$.

$G_1:$	$M_0 1 G$	$\otimes$	$\gamma_5 u_\lambda$
$G_2:$	$M_0\gamma_5 G$	$\otimes$	u_λ
$G_3:$	$M_0\gamma^\mu\vec{\tau}G$	$\otimes$	$\gamma_5\gamma_\mu\vec{\tau}u_\lambda$
$G_4:$	$M_0\gamma^\mu\gamma_5 G$	$\otimes$	$\gamma_\mu u_\lambda$
$G_5:$	$\gamma\cdot P\vec{\tau}G$	$\otimes$	$\gamma_5\vec{\tau}u_\lambda$
$G_6:$	$\gamma\cdot P\gamma_5 G$	$\otimes$	u_λ
$G_7:$	$M_0\sigma^{\mu\nu}\vec{\tau}G$	$\otimes$	$\gamma_5\sigma_{\mu\nu}\vec{\tau}u_\lambda$
$G_8:$	$i\sigma^{\mu\nu}P_\nu\vec{\tau}G$	$\otimes$	$\gamma_5\gamma_\mu\vec{\tau}u_\lambda$

TABLE II. Relativistic spin invariants for the nucleon resonance $N^*(1535)$, $|\left[(s_{12}\frac{1}{2})\frac{1}{2}1\right];\frac{1}{2}^{-}\lambda\rangle$.

1.	$M_0 1\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
2.	$M_0\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma_5\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
3.	$M_0\gamma^\mu G$	$\otimes$	$\gamma_\mu\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
4.	$M_0\gamma^\mu\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma_\mu\gamma_5\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
5.	$\gamma\cdot PG$	$\otimes$	$\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
6.	$\gamma\cdot P\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma_5\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
7.	$M_0\sigma^{\mu\nu}G$	$\otimes$	$\sigma_{\mu\nu}\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$
8.	$i\sigma^{\mu\nu}P_\nu G$	$\otimes$	$\gamma_\mu\gamma\cdot(\tilde{p}_1-\tilde{p}_2)u_\lambda$

TABLE III. Basis states for nucleon resonance $N^*(1520)$, $|\left[(s_{12}\frac{1}{2})\frac{1}{2}1\right];\frac{3}{2}^{-}\lambda\rangle$.

1.	$M_0 1\vec{\tau}G$	$\otimes$	$\gamma_5\vec{\tau}(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
2.	$M_0\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
3.	$M_0\gamma^\mu G$	$\otimes$	$\gamma_5\gamma_\mu(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
4.	$M_0\gamma^\mu\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma_\mu(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
5.	$\gamma\cdot P\vec{\tau}G$	$\otimes$	$\vec{\tau}\gamma_5(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
6.	$\gamma\cdot P\gamma_5\vec{\tau}G$	$\otimes$	$\vec{\tau}(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
7.	$M_0\sigma^{\lambda\rho}G$	$\otimes$	$\gamma_5\sigma_{\lambda\rho}(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$
8.	$i\sigma^{\lambda\rho}P_\rho G$	$\otimes$	$\gamma_5\gamma_\lambda(\tilde{p}_1-\tilde{p}_2)_\nu u^\nu_{\frac{3}{2}\lambda}$

TABLE IV. BW-basis for positive parity and $S_z > 0$.

$B_1 = U^\uparrow U^\uparrow U^\uparrow$	$B_9 = V^\uparrow U^\uparrow V^\uparrow$
$B_2 = U^\uparrow U^\uparrow U^\downarrow$	$B_{10} = V^\uparrow U^\uparrow V^\downarrow$
$B_3 = U^\uparrow U^\downarrow U^\uparrow$	$B_{11} = V^\uparrow U^\downarrow V^\uparrow$
$B_4 = U^\downarrow U^\uparrow U^\uparrow$	$B_{12} = V^\downarrow U^\uparrow V^\uparrow$
$B_5 = U^\uparrow V^\uparrow V^\uparrow$	$B_{13} = V^\uparrow V^\uparrow U^\uparrow$
$B_6 = U^\uparrow V^\uparrow V^\downarrow$	$B_{14} = V^\uparrow V^\uparrow U^\downarrow$
$B_7 = U^\uparrow V^\downarrow V^\uparrow$	$B_{15} = V^\uparrow V^\downarrow U^\uparrow$
$B_8 = U^\downarrow V^\uparrow V^\uparrow$	$B_{16} = V^\downarrow V^\uparrow U^\uparrow$

TABLE V. Symmetrized BW states for positive parity and $m_j > 0$.

dim[$SU(4)$]	symmetrized BW-states	$\chi^{RS}_{m_R m_S}$
20_S	$\xi_S \varphi_S$	S'_1, S'_2, S''_1, S''_2
	$\frac{1}{\sqrt{2}}(\xi_{M_S}\varphi_{M_S} + \xi_{M_A}\varphi_{M_A})$	$= S$
20_{M_S}	$\xi_S \varphi_{M_S}$	s_1, s_2
	$\xi_{M_S} \varphi_S$	s'_1, s'_2
	$-\frac{1}{\sqrt{2}}(\xi_{M_S}\varphi_{M_S} - \xi_{M_A}\varphi_{M_A})$	$= s_3$
20_{M_A}	$\xi_S \varphi_{M_A}$	a_1, a_2
	$\xi_{M_A} \varphi_S$	a'_1, a'_2
	$\frac{1}{\sqrt{2}}(\xi_{M_S}\varphi_{M_A} + \xi_{M_A}\varphi_{M_S})$	$= a_3$
4_A	$\frac{1}{\sqrt{2}}(\xi_{M_S}\varphi_{M_A} - \xi_{M_A}\varphi_{M_S})$	$= A$

TABLE VI. $SU(2)$ spin states for three quarks.

m_S	$+\frac{3}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
φ_S	$\|\uparrow\uparrow\uparrow\rangle$	$\frac{1}{\sqrt{3}}\|\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow + \uparrow\uparrow\downarrow\rangle$	$\frac{1}{\sqrt{3}}\|\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow\rangle$	$\|\downarrow\downarrow\downarrow\rangle$
φ_{M_S}		$\frac{1}{\sqrt{6}}\|\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow\rangle$	$-\frac{1}{\sqrt{6}}\|\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow\rangle$	
φ_{M_A}		$\frac{1}{\sqrt{2}}\|\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle$	$-\frac{1}{\sqrt{2}}\|\downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow\rangle$	

TABLE VII. $SU(2)$ R-spin states for three quarks. The respective parity is given by π.

m_R	$+3$	$+1$	-1	-3
π	$+$	$-$	$+$	$-$
ξ_S	$\|UUU\rangle$	$\frac{1}{\sqrt{3}}\|UVU+VUU+UUV\rangle$	$\frac{1}{\sqrt{3}}\|UVV+VUV+VVU\rangle$	$\|VVV\rangle$
ξ_{M_S}		$\frac{1}{\sqrt{6}}\|UVU+VUU-2UUV\rangle$	$-\frac{1}{\sqrt{6}}\|UVV+VUV-2VVU\rangle$	
ξ_{M_A}		$\frac{1}{\sqrt{2}}\|UVU-VUU\rangle$	$-\frac{1}{\sqrt{2}}\|VUV-UVV\rangle$	

TABLE VIII. Explicit form of the functions $S' \ldots A$ not given in Table V.

$S_1' = \xi_S' \varphi_S$	$S_2' = \xi_S \varphi_S$	$S_1'' = \xi_S' \varphi_S'$	$S_2'' = \xi_S \varphi_S'$
$s_1 = \xi_S' \varphi_{M_S}$	$s_2 = \xi_S \varphi_{M_S}$		
$s_1' = \xi_{M_S} \varphi_S$	$s_2' = \xi_{M_S} \varphi_S'$		
$a_1 = \xi_S' \varphi_{M_A}$	$a_2 = \xi_S \varphi_{M_A}$		
$a_1' = \xi_{M_A} \varphi_S$	$a_2' = \xi_{M_A} \varphi_S'$		

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